Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(99)00025-8

J. Appl. Maths Mechs, Vol. 63, No. 2, pp. 173–188, 1999

0021-8928/99/\$—see front matter

THE ROTATIONAL MOTIONS OF MECHANICAL SYSTEMS†

V. N. TKHAI

Moscow

(*Received* 10 *April* 1998)

Autonomous systems or systems that are periodic with respect to the independent variable t, specified on $\mathbb{R}^l \times \mathbb{T}^n$ ($\mathbb{T}^n$ is a torus of dimension n) are considered. In this paper $2\pi k$-periodic rotational motions (including oscillatory motions), closed on $\mathbb{R}^l \times \mathbb{T}^n$ (in a time $\Delta^t = 2\pi k$, $k \in \mathbb{N}$, for a system that is 2π-periodic in t) are investigated. It is shown that for such motions a theory analogous to the theory for oscillatory motions holds. In particular, Poincaré's theorem on the presence of a zero characteristic exponent in an autonomous system, the Andronov–Vitt theorem on the stability of rotational periodic motion of an autonomous system, and the theory of the continuation of rotational periodic motion with respect to a small parameter hold. The necessary and sufficient conditions for periodic rotational motion to exist are given for a reversible system, and a method is proposed for constructing all such motions. A detailed investigation is made of periodic rotational motions of a system, close to a conservative system with one degree of freedom. It is shown, in particular, that steady motions of an average system correspond in Volosov's method to exact periodic rotational motions. All $(2\pi k/|m|)$ periodic rotational motions of a conservative system ($k \in \mathbb{N}$, $m \in \mathbb{Z}\backslash\{0\}$) are conserved (in the sense of continuation with respect to a parameter) when small reversible perturbations, 2π-periodic with respect to t, act on it. It is show, in the problem of the motion of a satellite in the plane of the elliptic orbit under gravitational forces (the Beletskii problem), that additional perturbing factors have no effect on the qualitative conclusions regarding the existence of periodic rotational and oscillatory motions or on the stability of such motions. © 1999 Elsevier Science Ltd. All rights reserved.

In the problem of a mathematical pendulum, a remarkable relationship is observed between the oscillatory and rotational motions. These two qualitatively different forms of motion are also described mathematically by functions of different classes. However, there is a general property which unites the oscillatory and rotational motions: both are periodically repeating processes. In other words, the oscillatory and rotational motions are examples of periodic motion.

We usually mean by periodic motion the solution of the describing system of differential equations, defined by periodic functions. The periodic motion here is a closed integral curve in phase space.

In the problem of the mathematical pendulum the oscillatory and rotational motions are also represented by closed curves, if we change from the phase plane to a phase cylinder. On this cylinder the state of the system is defined by the angular coordinate (the angle of rotation of the pendulum) and the angular velocity, measured along the cylinder axis. The oscillatory motion on the phase cylinder is a special case of rotational motion, when the number of rotations of the representative point around the cylinder along the closed curve is zero.

For a closed curve on a phase cylinder the angle φ satisfies the condition

$$\varphi(t+\tau) = \varphi(t) + 2\pi m, \; m \in \mathbb{Z}$$

(τ is the time taken to make a complete circuit round the closed curve). Then $\varphi^{\cdot}(t)$ is a periodic function of t with period τ and

$$\varphi(t) = (2\pi m/\tau)t + \psi(t), \; \psi(t+\tau) = \psi(t)$$

so, on changing to a system of coordinates, rotating uniformly with angular velocity $2\pi m/\tau$, we have oscillatory motion.

This approach can be used to investigate rotational motions. However, some difficulties arise here due to the dependence of the period τ on the initial conditions. Another approach to investigating both oscillatory and rotational motions using a single theory is based on the fact that the corresponding integral curve is closed. Thus, for example, the rotations of a satellite in the plane of the elliptic orbit has been investigated [1, 2].‡ It turns out that the theory developed using the second approach is completely analogous to the well-developed theory of non-linear oscillations.

In this paper we show how, using the theory of non-linear oscillations, one can construct a theory of periodic motions, which enables both oscillatory and rotational motions to be investigated using a single approach. Rotational motions which, in a special case, degenerate into oscillatory motion are considered.

†*Prikl. Mat. Mekh.* Vol. 63, No. 2, pp. 179–195, 1999.

‡See also VARIN, V. P., Generalized periodic solutions of the equations of the oscillations of a satellite. Preprint No. 97, Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, 1997.

1. ROTATIONAL MOTIONS AND THEIR STABILITY

Consider the system

$$\mathbf{x}^{\cdot} = \mathbf{X}(\mathbf{x}, \mathbf{y}, t), \quad \mathbf{y}^{\cdot} = \mathbf{Y}(\mathbf{x}, \mathbf{y}, t) \tag{1.1}$$

(**x** is an l-vector and **y** is an n-vector) with right-hand sides that are 2π-periodic with respect to t and with respect to the vector **y**. System (1.1) is thereby specified on $\mathbb{R}^l \times \mathbb{T}^n$, where $\mathbb{T}^n$ is a torus of dimension n. The solution of this system $\mathbf{x} = \varphi(t)$, $\mathbf{y} = \psi(t)$ will be called a $2\pi k$-periodic motion if

$$\varphi(t + 2T^*) = \varphi(t), \ \psi(t + 2T^*) = \psi(t) + 2\pi\mathbf{m}, \ \mathbf{m} \in \mathbb{Z}^n, T^* = \pi k \tag{1.2}$$

In this case the integral curve on $\mathbb{R}^l \times \mathbb{T}^n$ is closed after a time $\Delta t = 2\pi k$. Conversely, if the integral curve on $\mathbb{R}^l \times \mathbb{T}^n$ is closed after a time $\Delta t = 2\pi k$, the functions $\varphi(t)$, $\psi(t)$ satisfy condition (1.2).

Hence, conditions (1.2) completely define $2\pi k$-periodic motion and give the necessary and sufficient conditions for such motion to exist

$$\varphi(t_0 + 2T^*) = \varphi(t_0), \ \psi(t_0 + 2T^*) = \psi(t_0) + 2\pi\mathbf{m}, \ \mathbf{m} \in \mathbb{Z}^n, T^* = \pi k$$

(t_0 is the initial instant of time).

Solution (1.2) describes both oscillatory motion ($\mathbf{m} = \mathbf{0}$) and rotational motion ($\mathbf{m} \neq \mathbf{0}$). When ($\mathbf{m} \neq \mathbf{0}$) solution (1.2) is not $2\pi k$-periodic in the usual sense of the word. Nevertheless, taking the above explanations into account, everywhere in this paper we will consider $2\pi k$-periodic motions of the form (1.2). In the case of autonomous system (1.1) we will investigate $2T^*$-periodic solutions, closed on $\mathbb{R}^l \times \mathbb{T}^n$ after a time $\Delta t = 2T^*$. Finally, note that the cases when $l = 0$ or $n = 0$ are also covered by the investigations below.

Suppose autonomous system (1.2) allows of $2T^*$-periodic motion (1.2). Then the equations in variations for (1.2) have the particular solution $\varphi^{\cdot}(t)$, $\psi^{\cdot}(t)$, which is $2T^*$-periodic in the usual sense of the word

$$\varphi^{\cdot}(t + 2T^*) = \varphi^{\cdot}(t), \ \psi^{\cdot}(t + 2T^*) = \psi^{\cdot}(t)$$

where the equations in variations themselves represent a system of linear differential equations with coefficients that are $2T^*$-periodic in t. Consequently, Poincaré's theorem holds: the characteristic equation has at least one root equal to unity.

The equations of perturbed motion for $2T^*$-periodic motion (1.2) are $2T^*$-periodic in t. When the remaining $l + n - 1$ roots have moduli less than unity, these equations allow of a single-parameter family of periodic solutions. Hence the following assertion holds.

Theorem 1 (the Andronov–Vitt theorem [4]). $2T^*$-periodic motion (1.2) of autonomous system (1.1) is Lyapunov stable if $l + n - 1$ roots of the characteristic equation have moduli less than unity.

The proof repeats word for word the proof given in [5] for the case of a solution that is periodic in the usual meaning of the word.

The generalization [5] of the Andronov–Vitt theorem also holds.

If system (1.1) allows of a p-family on $2\pi k$-periodic rotational motions ($p \leq l + n$), each of the solutions of this family is Lyapunov stable, if the characteristic equation has $l + n - p$ characteristic exponents that are less than unity in modulus.

2. THE PROBLEM OF THE CONTINUATION OF ROTATIONAL MOTION WITH RESPECT TO A PARAMETER

Consider the sufficiently smooth autonomous of 2π-periodic system

$$\begin{aligned} \mathbf{x}^{\cdot} &= \mathbf{X}(\mathbf{x}, \mathbf{y}, t) + \mu\mathbf{X}_1(\mu, \mathbf{x}, \mathbf{y}, t) \\ \mathbf{y}^{\cdot} &= \mathbf{Y}(\mathbf{x}, \mathbf{y}, t) + \mu\mathbf{Y}_1(\mu, \mathbf{x}, \mathbf{y}, t); \ \mathbf{x} \in \mathbb{R}^l, \mathbf{y} \in \mathbb{T}^n \end{aligned} \tag{2.1}$$

(μ is a parameter), specified on $\mathbb{R}^l \times \mathbb{T}^n$. We will assume that when $\mu = 0$ the generating system (1.1) obtained allows of $2T^*$-periodic motion of the form (1.2), where $T^* = \pi/k$ in the case of system (2.1),

2π-periodic with respect to t. We will investigate the problem of the existence in (2.1) when $\mu \neq 0$ of $2T$-periodic motion, which converts into motion (1.2) as $\mu \to 0$.

The necessary and sufficient conditions for $2T$-periodic motion to exist have the form

$$\mathbf{x}(\mu, \mathbf{x}^0, \mathbf{y}^0, 2T) = \mathbf{x}^0$$
$$\mathbf{y}(\mu, \mathbf{x}^0, \mathbf{y}^0, 2T) = \mathbf{y}^0 + 2\pi\mathbf{m}, \ \mathbf{m} \in \mathbb{Z}^n \tag{2.2}$$

where $\mathbf{x}(\mu, \mathbf{x}^0, \mathbf{y}^0, t)$, $\mathbf{y}(\mu, \mathbf{x}^0, \mathbf{y}^0, t)$ is the general solution of system (2.1) with initial point (x^0, y^0) when $t = 0$. System (2.1) has a solution when $\mu = 0$: $\mathbf{x}^0 = \varphi(0)$, $\mathbf{y}^0 = \mathbf{y}^* = \psi(0)$. Hence, it follows from the theory of implicit functions that system (2.2) is consistent for sufficiently small $|\mu| \neq 0$ if the following condition is satisfied:

in the case when system (2.1) is 2π-periodic in t

$$\operatorname{rank}\left\| \begin{matrix} \dfrac{\partial \mathbf{x}(0,\mathbf{x}^0,\mathbf{y}^0,2\pi k)}{\partial \mathbf{x}^0} - \mathbf{I}_l & \dfrac{\partial \mathbf{x}(0,\mathbf{x}^0,\mathbf{y}^0,2\pi k)}{\partial \mathbf{y}^0} \\ \dfrac{\partial \mathbf{y}(0,\mathbf{x}^0,\mathbf{y}^0,2\pi k)}{\partial \mathbf{x}^0} & \dfrac{\partial \mathbf{y}(0,\mathbf{x}^0,\mathbf{y}^0,2\pi k)}{\partial \mathbf{y}^0} - \mathbf{I}_n \end{matrix} \right\|_* = l + n \tag{2.3}$$

in the case of an autonomous system

$$\operatorname{rank}\left\| \begin{matrix} \dfrac{\partial \mathbf{x}(0,\mathbf{x}^0,\mathbf{y}^0,2T)}{\partial \mathbf{x}^0} - \mathbf{I}_l & \dfrac{\partial \mathbf{x}(0,\mathbf{x}^0,\mathbf{y}^0,2T)}{\partial \mathbf{y}^0} & \mathbf{X}(\varphi(2T),\psi(2T),2T) \\ \dfrac{\partial \mathbf{y}(0,\mathbf{x}^0,\mathbf{y}^0,2T)}{\partial \mathbf{x}^0} & \dfrac{\partial \mathbf{y}(0,\mathbf{x}^0,\mathbf{y}^0,2T)}{\partial \mathbf{y}^0} - \mathbf{I}_n & \mathbf{Y}(\varphi(2T),\psi(2T),2T) \end{matrix} \right\|_* = l + n \tag{2.4}$$

($\mathbf{I}_j$ is the identity j-matrix).

The calculations in (2.3) and (2.4) are carried out for $\mathbf{x}^0 = \mathbf{x}^*$, $\mathbf{y}^0 = \mathbf{y}^*$, $T = T^*$.

By analogy with the case of motion that is periodic in the usual meaning of the word ($m = 0$), when condition (2.3) or (2.4) is satisfied we have the Poincaré-isolated case [6].

Theorem 2. In the Poincaré-isolated case, system (2.1), for sufficiently small $|\mu| \neq 0$, has a unique $2T$-periodic motion ($T = \pi k$ in the case of a 2π-periodic system), which converts into motion (1.2) as $\mu \to 0$.

Notes. 1. Condition (2.3) or (2.4) is checked by calculating the roots of the characteristic equation of the system in variations.

2. A complete analogy follows from Theorem 2 in problems of the continuation with respect to a parameter of oscillatory motion ($m = 0$) and rotational motion ($\mathbf{m} \neq \mathbf{0}$), where, in the isolated case, both problems are solved by Theorem 2.

3. The non-isolated cases in the theory of rotational motions are also investigated as in the theory of oscillatory motions.

3. A SYSTEM OF STANDARD FORM

We will assume that $\mathbf{X}(\mathbf{x}, \mathbf{y}, t) = \mathbf{0}$ in system (2.1). We then obtain the following system of standard form, which is important in applications

$$\mathbf{x}^{\cdot} = \mu\mathbf{X}(\mu, \mathbf{x}, \mathbf{y}, t)$$
$$\mathbf{y}^{\cdot} = \boldsymbol{\omega}(\mathbf{x}, \mathbf{y}, t) + \mu\mathbf{Y}(\mu, \mathbf{x}, \mathbf{y}, t) \tag{3.1}$$

We will first investigate periodic system (3.1). Equations (2.2) have the form

$$\mathbf{x}^0 + \mu\mathbf{x}_1(\mu, \mathbf{x}^0, \mathbf{y}^0, 2\pi k) = \mathbf{x}^0$$
$$\psi(\mathbf{x}^0, \mathbf{y}^0, 2\pi k) + \mu\mathbf{y}_1(\mu, \mathbf{x}^0, \mathbf{y}^0, 2\pi k) = \mathbf{y}^0 + 2\pi\mathbf{m} \tag{3.2}$$

where

$$\mathbf{x} = \mathbf{x}^0(\text{const}),\ \mathbf{y} = \psi(\mathbf{x}^0, \mathbf{y}^0, t)$$

is a solution of the generating system

$$\mathbf{x}^{\cdot} = 0,\ \mathbf{y}^{\cdot} = \omega(\mathbf{x}, \mathbf{y}, t) \tag{3.3}$$

with initial point $(\mathbf{x}^0, \mathbf{y}^0)$. System (3.2) has a unique solution for sufficiently small $|\mu|$ if the system of functional equations

$$\mathbf{I}(\mathbf{x}^0, \mathbf{y}^0) \equiv \int_0^{2\pi k} \mathbf{X}(0, \mathbf{x}^0, \psi(\mathbf{x}^0, \mathbf{y}^0, t))dt = \mathbf{0} \tag{3.4}$$
$$\psi(\mathbf{x}^0, \mathbf{y}^0, 2\pi k) = \mathbf{y}^0 + 2\pi\mathbf{m}$$

has a simple root $(\mathbf{x}^*, \mathbf{y}^*)$. This root will obviously be simple when the following condition is satisfied

$$\operatorname{rank}\left\| \begin{matrix} \dfrac{\partial \mathbf{I}}{\partial \mathbf{x}^0} & \dfrac{\partial \mathbf{I}}{\partial \mathbf{y}^0} \\ \dfrac{\partial \psi}{\partial \mathbf{x}^0} & \dfrac{\partial \psi}{\partial \mathbf{y}^0} - \mathbf{I}_n \end{matrix} \right\|_* = l + n \tag{3.5}$$

(the calculations are carried out for the point $(\mathbf{x}^*, \mathbf{y}^*)$.

The second equation in (3.4) reflects the fact that the $2\pi k$-periodicity of the motion of generating system (3.3) also consists of n scalar equations in the $l + n$ unknowns $x_1^0, \ldots, x_l^0, y_1^0, \ldots, y_n^0$. Consequently, its solution contains l arbitrary parameters and defines an l-family of $2\pi k$-periodic motions of the generating system. Various interesting cases are possible here.

1. *A family of "amplitude"* $\mathbf{x}^0$. Suppose, when $\mathbf{y}^0 = \mathbf{y}^*$, the second equation in (3.4) becomes an identity in $\mathbf{x}^0$. We then have $(\partial\psi/\partial\mathbf{x}^0)_* \equiv \mathbf{0}$ in (3.5) and the sufficient conditions for system (3.2) to be solvable are as follows: (a) the equations in variations

$$d\mathbf{z}/dt = \partial\omega/\partial\mathbf{y}\,|_*\, \mathbf{z}$$

for the second equation in (3.3), which are constant for $2\pi k$-periodic motion $\mathbf{x} = \mathbf{x}^*$, $\mathbf{y} = \psi(\mathbf{x}^*, \mathbf{y}^*, t)$ of generating system (3.3), have no $2\pi k$-periodic solution
(b) the amplitude equation

$$\int_0^{2\pi k} \mathbf{X}(0, \mathbf{x}^0, \psi(\mathbf{x}^0, \mathbf{y}^*, t), t)dt = \mathbf{0}$$

has a simple root $\mathbf{x}^0 = \mathbf{x}^*$.

When these conditions are satisfied, system (3.1) has the following unique $2\pi k$-periodic solution for sufficiently small $|\mu| \neq 0$

$$\mathbf{x} = \mathbf{x}^* + \mu\int^t \mathbf{X}(0, \mathbf{x}^*, \psi(\mathbf{x}^*, \mathbf{y}^*, t), t)dt + o(\mu),\ \ \mathbf{y} = \psi(\mathbf{x}^*, \mathbf{y}^*, t) + \mu\psi_1(t) + o(\mu) \tag{3.6}$$

where $\psi_1(t)$ is a particular $2\pi k$-periodic solution of the equation

$$d\mathbf{z}/dt = \partial\omega/\partial\mathbf{y}\,|_*\, \mathbf{z} + \mathbf{Y}(0, \mathbf{x}^*, \mathbf{y}^*, \psi(\mathbf{x}^*, \mathbf{y}^*, t), t)$$

This solution exists and is unique by virtue of condition (a).

2. *A family of initial "angle"* $\mathbf{y}^0$. We will first assume that

$$l \geq n, \mathbf{x}^0 = \mathbf{x}^0(\alpha_1, \ldots, \alpha_{l-n}, \beta_1, \ldots, \beta_n)$$

and the second equation in (3.4) is satisfied when $\boldsymbol{\beta} = \boldsymbol{\beta}^*$ identically with respect to $\mathbf{y}^0$ and $\boldsymbol{\alpha}$. Then, with this value of $\boldsymbol{\beta}$ we have

$$r = \operatorname{rank}(\partial\boldsymbol{\psi}/\partial\mathbf{x}^0)_* \leq n, \ (\partial\boldsymbol{\psi}/\partial\mathbf{y}^0 - \mathbf{I}_n)_* \equiv \mathbf{0}$$

If $r = n$ in this case, then to each root $(\boldsymbol{\alpha}^*, \mathbf{y}^*)$ of the amplitude equation

$$\int_0^{2\pi k} \mathbf{X}(0, \mathbf{x}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}^*), \boldsymbol{\psi}(\mathbf{x}^0(\boldsymbol{\alpha}, \boldsymbol{\beta}^*), \mathbf{y}^0, t), t)dt = \mathbf{0}$$

for small $|\mu| \neq 0$ there corresponds a unique $2\pi k$-periodic motion of the form (3.6). When $\beta = (x_1^0, \ldots, x_n^0)^T$ (T indicates transposition) the condition $r = n$ is verified by integrating n systems of linear inhomogeneous equations

$$z_s^{\cdot} = \sum_{j=1}^{n} p_{sj}(t) z_j + \left.\frac{\partial \omega_s}{\partial x_k}\right|_*, \ s, k = 1, \ldots, n; \ \mathbf{P} = \| p_{sj} \| = \left.\frac{\partial \boldsymbol{\omega}}{\partial \mathbf{y}}\right|_*$$

with zero initial conditions. When $\mathbf{P} \equiv \mathbf{0}$ we have $r = n$ if

$$\det \left\| \int_0^{2\pi k} \left.\frac{\partial \omega_s}{\partial x_k}\right|_* dt \right\| \neq 0 \tag{3.7}$$

Suppose now that $l < n$ and the second group of equations in (3.4) is satisfied for arbitrary $y_1^0, \ldots, y_l^0$ and fixed $\mathbf{x}^0 = \mathbf{x}^*, y_{l+1}^0 = y_{l+1}^*, \ldots, y_n^0 = y_n^*$. Then, to determine $y_1^0, \ldots, y_l^0$ we have a system of l amplitude equations

$$\int_0^{2\pi k} \mathbf{X}(0, \mathbf{x}^*, \boldsymbol{\psi}(\mathbf{x}^*, y_1^0, \ldots, y_l^0, y_{l+1}^*, \ldots, y_n^*, t), t)dt = \mathbf{0}$$

To each simple root $(y_1^*, \ldots, y_l^*)$ of this system, which satisfied the condition

$$\operatorname{rank}\|(\partial\boldsymbol{\psi}/\partial\mathbf{x}^0)_*, (\partial\boldsymbol{\psi}/\partial\mathbf{y}^0)_* - \mathbf{I}_n\| = n$$

(the asterisk denotes a calculation with $\mathbf{x}^0 = \mathbf{x}^*$, $\mathbf{y}^0 = \mathbf{y}^*$, $t = 2\pi k$) when $\mu \neq 0$, there corresponds a unique $2\pi k$-periodic motion of Eq. (3.1).

When $l = 0$ there is no amplitude equation and the condition for $2\pi k$-periodic motion to exist when $\mu \neq 0$ is the inequality

$$\det\|\partial\boldsymbol{\psi}/\partial\mathbf{y}^0 - \mathbf{I}_n\|_* \neq 0$$

if $\mathbf{y}^*$ is the root of the second equation of (3.2).

3. *The frequency depends only on the "amplitude"* $\mathbf{x}^0$. We will consider an important special case when the function $\boldsymbol{\omega}$ depends only on the vector $\mathbf{x}$. Here the general solution of generating system (3.3) has the form

$$\mathbf{x} = \mathbf{x}^0, \ \mathbf{y} = \boldsymbol{\omega}(\mathbf{x}^0)t + \mathbf{y}^0 \tag{3.8}$$

Formulae (3.8) obviously describe $2\pi k$-periodic motion if $\boldsymbol{\omega}(\mathbf{x}^0) = \mathbf{m}/k$ ($\mathbf{m} \in \mathbb{Z}^n$, $k \in \mathbb{N}$), and when $\mathbf{m} = 0$ we have a constant solution. When $\mathbf{m} \neq \mathbf{0}$ the motion is rotational: the "amplitude" $\mathbf{x}$ is constant, while the "angle" $\mathbf{y}$ changes by $2\pi\mathbf{m}$ when the time increases by $2\pi k$.

The case considered is a special case of paragraph 2 above. Hence, the sufficient condition for a unique $2\pi k$-periodic motion to exist when $\mu \neq 0$ is that a simple root $(\mathbf{x}^*, \mathbf{y}^*)$ of the system of functional equations

$$\boldsymbol{\omega}(\mathbf{x}^0)=\frac{\mathbf{m}}{k},\quad \int_0^{2\pi k}\mathbf{X}(0,\mathbf{x}^0,\boldsymbol{\omega}(\mathbf{x}^0)t+\mathbf{y}^0,t)dt=\mathbf{0} \tag{3.9}$$

should exist, where condition (3.7) takes the form

$$\operatorname{rank}\|\partial\boldsymbol{\omega}(\mathbf{x}^0)/\partial\mathbf{x}^0\|_*=n \tag{3.10}$$

Suppose $(\mathbf{x}^*, \mathbf{y}^*)$ is a simple root of system (3.9). System (3.2) can then be written in the form

$$\mathbf{x}_1(0,\mathbf{x}^0,\mathbf{y}^0,2\pi k)+\mathrm{o}(\mu)=\mathbf{0} \tag{3.11}$$

$$\frac{\partial\boldsymbol{\omega}(\mathbf{x}^*)}{\partial\mathbf{x}^*}\Delta\mathbf{x}+\frac{\mu}{2\pi k}\int_0^{2\pi k}\mathbf{Y}(0,\mathbf{x}^*,\boldsymbol{\omega}(\mathbf{x}^*)t+\mathbf{y}^*,t)dt+o(\mu)+o(\Delta x)=0$$

$$(\Delta\mathbf{x}=\mathbf{x}^0-\mathbf{x}^*)$$

Put $\Delta x = \mu\xi$. Then, when condition (3.10) is satisfied, from the second equation in (3.11) we can determine the vector ξ, that contains $l - n$ arbitrary components. These components, and also $\Delta\mathbf{y} = \mathbf{y}^0 - \mathbf{y}^* = \mu\boldsymbol{\eta}$, are found from the first equation in (3.11). As a result, the solution which describes $2\pi k$-periodic motion has the form

$$\mathbf{x}=\mathbf{x}^*+\mu[\boldsymbol{\xi}+\int_0^t\mathbf{X}(0,\mathbf{x}^*,\boldsymbol{\omega}(\mathbf{x}^*)t+\mathbf{y}^*,t)dt]+o(\mu)$$

$$\mathbf{y}=\mathbf{y}^*+(\mathbf{m}/k)t+\mu[\boldsymbol{\eta}+\frac{\partial\boldsymbol{\omega}(\mathbf{x}^*)}{\partial\mathbf{x}^*}\boldsymbol{\xi}t+\int_0^t\mathbf{Y}(0,\mathbf{x}^*,\boldsymbol{\omega}(\mathbf{x}^*)t+\mathbf{y}^*,t)dt]+o(\mu)$$

In the degenerate case, when we have $\boldsymbol{\omega}(\mathbf{x}^*) = \mathbf{m}/k$, $r < n$ for certain $\mathbf{x}^*$, we need to make the replacement $\mathbf{x} = \mathbf{x}^* + \mathbf{u}$, $\mathbf{y} = (\mathbf{m}/k)t + \mathbf{v}$. We then obtain, in $\mathbf{u}$, $\mathbf{v}$ variables, the problem of oscillatory motions in a system of standard form. This problem was considered previously in [7].

4. *An autonomous system*. Two approaches are possible when investigating an autonomous system: (1) analysis of a system of the form (3.2), in which the period $2\pi k$ is replaced by $2T$, and the conditions for this system to be solvable are derived, and (2) reduction to a periodic system.

We will illustrate the second approach when the function $\boldsymbol{\omega}$ depends only on the variable $\mathbf{x}$. In rotational motion of the generating system we have $\boldsymbol{\omega}(\mathbf{x}^0) \neq \mathbf{0}$. The inequality $\boldsymbol{\omega}(\mathbf{x}) \neq \mathbf{0}$ arises in a finite time interval also for small $|\mu| \neq 0$. We will assume that the last component $\omega_n(\mathbf{x})$ of the vector $\boldsymbol{\omega}(\mathbf{x})$ is non-zero. Then, we can introduce a new independent variable y_n in the problem of periodic motions. By defining a unique $2\pi k$-periodic motion

$$\mathbf{x}=\mathbf{x}(y_n),\ y_1=y_1(y_n),\ \ldots,\ y_{n-1}=y_{n-1}(y_n)$$

we get a single autonomous differential equation for determining $y_n(t)$.

4. PERIODIC MOTIONS OF REVERSIBLE SYSTEMS

Consider the following reversible system, 2π-periodic in t

$$\mathbf{u}^{\cdot}=\mathbf{U}(\mathbf{u},\mathbf{v},t),\ \mathbf{v}^{\cdot}=\mathbf{V}(\mathbf{u},\mathbf{v},t) \tag{4.1}$$

which is invariant under the transformation $(t, \mathbf{u}, \mathbf{v}) \to (-t, \mathbf{u}, -\mathbf{v})$. We will assume that system (4.1) is 2π-periodic with respect to all or some of the components of the vector $\mathbf{v}$ and is specified on $\mathbb{R}^p \times \mathbb{T}^q$. For simplicity we will consider below the case when $p = l$ and $q = n$ (l and n are the dimensions of the vectors $\mathbf{u}$ and $\mathbf{v}$, respectively); the general case when $p \geqslant l$, $q \geqslant n$, is easily considered from the one described.

With these assumptions the functions $\mathbf{U}(\mathbf{u}, \mathbf{v}, t)$ and $\mathbf{V}(\mathbf{u}, \mathbf{v}, t)$ will be odd and even respectively with respect to the set of variables $(t, \mathbf{v})$. In addition, these functions are 2π-periodic with respect to $(t, \mathbf{v})$

and are represented by Fourier sine and cosine series, respectively. Obviously this representation is retained when $(t, \mathbf{u}, \mathbf{v})$ is replaced by $(t + \pi\alpha, \mathbf{u}, \mathbf{v} + \pi\boldsymbol{\beta})$; $\alpha \in \mathbb{Z}$, $\boldsymbol{\beta} \in \mathbb{Z}^n$. Then the transformed system is also invariant under the replacement $(t, \mathbf{u}, \mathbf{v}) \to (-t, \mathbf{u}, -\mathbf{v})$. Hence it follows that the two transformations $(t, \mathbf{u}, \mathbf{v}) \to (t + \pi\alpha, \mathbf{u}, \mathbf{v} + \pi\boldsymbol{\beta})$ and $(t, \mathbf{u}, \mathbf{v}) \to (-t + \pi\alpha, \mathbf{u}, -\mathbf{v} + \pi\boldsymbol{\beta})$ of the initial system (4.1), which lead to the same system, exist.

The transformation $(t + \pi\alpha, \mathbf{u}, \mathbf{v} + \pi\boldsymbol{\beta}) \to (-t + \pi\alpha, \mathbf{u}, -\mathbf{v} + \pi\boldsymbol{\beta})$ has the fixed set

$$\mathbf{M} = \{t, \mathbf{u}, \mathbf{v}: \sin t = 0, \sin v_s = 0 \ (s = 1, ..., n)\}$$

which we will call a fixed set of the reversible system (4.1), specified on $\mathbb{R}^l \times \mathbb{T}^n$.

Suppose $\mathbf{u}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$, $\mathbf{v}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$ is a solution of system (4.1) with initial point $(\mathbf{u}^0, \mathbf{v}^0)$ when $t = t^0$. In (4.1) we then have two solutions simultaneously

$$\mathbf{u} = \mathbf{u}(\mathbf{u}^0, \pm\mathbf{b} + \pi\boldsymbol{\beta}, \pm a + \pi\alpha, \pm a + \pi\alpha \pm t)$$
$$\mathbf{v} = \pm\mathbf{v}(\mathbf{u}^0, \pm\mathbf{b} + \pi\boldsymbol{\beta}, \pm a + \pi\alpha, \pm a + \pi\alpha \pm t) \mp \pi\boldsymbol{\beta}$$
$$a \in \mathbb{R}, \mathbf{b} \in \mathbb{R}^n$$

These solutions are symmetric with respect to the set $\mathbf{M}$. If $a = 0$, $\mathbf{b} = \mathbf{0}$, the two solutions are identical and represent a solution that is symmetric with respect to the fixed set $\mathbf{M}$.

For the symmetric solution we have

$$\begin{aligned} \mathbf{u}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, -t + \pi\alpha) &= \mathbf{u}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, t + \pi\alpha) \\ \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, -t + \pi\alpha) &= -\mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, t + \pi\alpha) + 2\pi\boldsymbol{\beta} \end{aligned} \tag{4.2}$$

This solution will be $2\pi k$-periodic ($k \in \mathbb{N}$), if

$$\begin{aligned} \mathbf{u}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi(2k + \alpha)) &= \mathbf{u}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi\alpha) = \mathbf{u}^0 \\ \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi(2k + \alpha)) &= \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi\alpha) + 2\pi\mathbf{m}, \ \mathbf{m} \in \mathbb{Z}^n \end{aligned} \tag{4.3}$$

Then, on $\mathbb{R}^l \times \mathbb{T}^n$ the symmetrical, $2\pi k$-periodic motion is closed after k rotations with respect to t and m_s rotations with respect to $v_s (s = 1, \dots, n)$; the sign of m_s indicates the direction of the motion with respect to $\mathbf{v}_s$. When $m = 0$ we have oscillatory motion and when $\mathbf{m} \neq \mathbf{0}$ the motion will be rotational.

The following proposition is fundamental for investigating symmetrical periodic motions of system (4.1).

Theorem 3. In order that the motion $\mathbf{u}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$, $\mathbf{v}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$ of system (4.1) should be symmetrical and $2\pi k$-periodic, it is necessary and sufficient for the following conditions to be satisfied

$$t^0 = \pi\alpha, \ \mathbf{v}^0 = \pi\boldsymbol{\beta}, \ \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi(k + \alpha)) = \pi(\mathbf{m} + \boldsymbol{\beta}), \ \mathbf{m} \in \mathbb{Z}^n \tag{4.4}$$

which indicates intersection of the fixed set M at the instants of time $\pi\alpha$ and $\pi(k + \alpha)$.

Proof. The symmetry of the solution follows from the conditions $t^0 = \pi\alpha$, $\mathbf{v}^0 = \pi\boldsymbol{\beta}$. Hence equalities (4.2) hold. We will put

$$\mathbf{u}_\pm^* = \mathbf{u}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi(\pm k + \alpha)), \ \mathbf{v}_\pm^* = \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, \pi\alpha, \pi(\pm k + \alpha))$$

Substituting $t = -\pi k$ into (4.2) we have

$$\mathbf{v}_+^* = -\mathbf{v}_-^* + 2\pi\boldsymbol{\beta} \tag{4.5}$$

The motion is $2\pi k$-periodic, and hence we obtain from (4.3)

$$\mathbf{v}_+^* = \mathbf{v}_-^* + 2\pi\mathbf{m} \tag{4.6}$$

Equalities (4.5) and (4.6) lead immediately to condition (4.4).

We will prove the sufficiency. We have

$$\mathbf{u}(\mathbf{u}^0,\pi\boldsymbol{\beta},\pi\alpha,\pi(2k+\alpha))=\mathbf{u}(\mathbf{u}_+^*,\pi(\mathbf{m}+\boldsymbol{\beta}),\pi(k+\alpha),\pi(2k+\alpha))\overset{(4.2)}{=}$$
$$=\mathbf{u}(\mathbf{u}_+^*,\pi(\mathbf{m}+\boldsymbol{\beta}),\pi(k+\alpha),\pi\alpha)=\mathbf{u}(\mathbf{u}^0,\pi\boldsymbol{\beta},\pi\alpha,\pi\alpha)=\mathbf{u}^0$$
$$\mathbf{v}(\mathbf{u}^0,\pi\boldsymbol{\beta},\pi\alpha,\pi(2k+\alpha))=\mathbf{v}(\mathbf{u}_+^*,\pi(\mathbf{m}+\boldsymbol{\beta}),\pi(k+\alpha),\pi(2k+\alpha))\overset{(4.2)}{=}$$
$$=-\mathbf{v}(\mathbf{u}_+^*,\pi(\mathbf{m}+\boldsymbol{\beta}),\pi(k+\alpha),\pi\alpha)+2\pi(\mathbf{m}+\boldsymbol{\beta})=-\mathbf{v}(\mathbf{u}^0,\pi\boldsymbol{\beta},\pi\alpha,\pi\alpha)+2\pi(\mathbf{m}+\boldsymbol{\beta})=$$
$$=\pi(2\mathbf{m}+\boldsymbol{\beta})$$

(the number of the formula above the equality sign indicates a transition using formula (4.2)).

Figure 1 illustrates Theorem 3. The continuous curve represents a portion of the trajectory when t changes from $\pi\alpha$ to $\pi(k+\alpha)$, and the dashed curve represents a portion of the trajectory when t changes from $\pi(k+\alpha)$ to $\pi(2k+\alpha)$.

In an autonomous reversible system, specified on $\mathbb{R}^l \times \mathbb{T}^n$, the fixed set has the form $\mathbf{M} = \{\mathbf{u}, \mathbf{v}: \sin v_s = 0\ (s = 1, \dots, n)\}$.

Theorem 4. In order for the motion $\mathbf{u}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$, $\mathbf{v}(\mathbf{u}^0, \mathbf{v}^0, t^0, t)$ with initial point $(\mathbf{u}^0, \mathbf{v}^0)$ at $t = t^0$ of the autonomous reversible system

$$\mathbf{u}^\cdot = \mathbf{U}(\mathbf{u}, \mathbf{v}),\ \mathbf{v}^\cdot = \mathbf{V}(\mathbf{u}, \mathbf{v});\ \mathbf{U}(\mathbf{u}, -\mathbf{v}), = -\mathbf{U}(\mathbf{u}, \mathbf{v}),\ \mathbf{V}(\mathbf{u}, -\mathbf{v}) = \mathbf{V}(\mathbf{u}, \mathbf{v}) \tag{4.7}$$

specified on $\mathbb{R}^l \times \mathbb{T}^n$ to be symmetrical with respect to the set $\mathbf{M}$ and $2T$-periodic, it is necessary and sufficient for the following conditions to be satisfied

$$\mathbf{v}^0 = \pi\boldsymbol{\beta},\ \mathbf{v}(\mathbf{u}^0, \pi\boldsymbol{\beta}, t^0, t^0 + T) = \pi(\mathbf{m} + \boldsymbol{\beta}),\ \boldsymbol{\beta}, \mathbf{m} \in \mathbb{Z}^n \tag{4.8}$$

or, which is the same thing, the existence of at least two common points of the solution considered and of the fixed set $\mathbf{M}$.

Notes. 1. For an autonomous or periodic system, specified on $\mathbb{R}^{l+n}$, in Theorems 3 and 4 it is necessary to take $\mathbf{m} = \mathbf{0}$ (the Heinbockel–Struble theorem [8]). When investigating oscillatory motions ($\mathbf{m} = \mathbf{0}$) Theorems 3 and 4 are applicable irrespective of the periodicity with respect to $\mathbf{v}$ of system (4.1) or (4.7).

2. If system (4.1) or (4.7) is 2π-periodic only with respect to the components $v_{s+1}, \dots, v_n$ of the vector $\mathbf{v}$, then, when, using Theorems 3 and 4, it is necessary to assume $m_1 = 0, \dots, m_s = 0$, and when determining the fixed set one must put $v_1 = 0, \dots, v_s = 0$.

From Theorems 3 and 4 we obtain a method of constructing all symmetrical periodic motions of the reversible system on $\mathbb{R}^l \times \mathbb{T}^n$.

Theorem 5. Suppose $\mathbf{M}^\tau$ is the image of the fixed set $\mathbf{M}$ of the reversible system (4.1) or (4.7), obtained when t changes from t^0 to $t^0 + \tau$ ($t^0 = \pi\alpha$, $\alpha \in \mathbb{Z}$, for a system that is 2π-periodic with respect

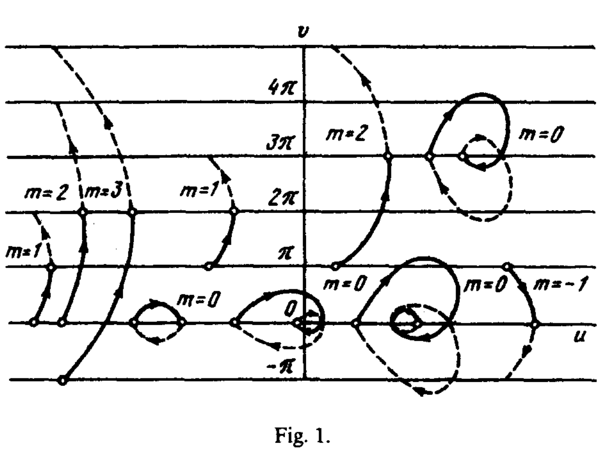

Fig. 1.

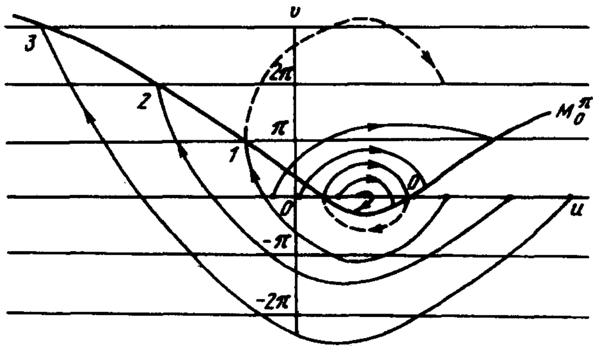

Fig. 2.

to t). Then, the set $\mathbf{M} \cap \mathbf{M}^\tau$ contains all points which, when $t = t^0 + \tau$, belong to the symmetrical, $(2\tau/\nu)$-periodic motions ($\nu = 1, 2, \ldots$) on $\mathbb{R}^l \times \mathbb{T}^n$, where $\tau = \pi k (k \in \mathbb{N})$, $\nu = 1$ for 2π-periodic system (4.1).

Note. The theorem was obtained previously [9] for the case of oscillatory motions.

Theorem 5 gives the correct solution of the problem of the numerical construction of all symmetrical periodic motions of the reversible system when $n = 1$. It is important that, as a result of a numerical investigation, an exact qualitative result should be derived regarding the existence of the required motion. The initial point for such motion can be calculated with the necessary accuracy. The situation here is analogous to that which exists for oscillatory motions [9].

Consider the 2π-periodic system (4.1) with $n = 1$, in which $\mathbf{M}_0^\pi$ is the image of the set $\mathbf{M}_0 = \{\mathbf{u}, \upsilon : \upsilon = 0\}$ when t changes from $\pi\alpha$ to $\pi(k + \alpha)$ (Fig. 2). Then the points of intersection of $\mathbf{M}_0^\pi$ and $\mathbf{M}^* = \{\mathbf{u}, \upsilon : \sin \upsilon = 0\}$ belong to symmetrical periodic motions. Here both oscillatory motions ($m = 0$) and different 2π-periodic rotational motions ($m = 1, 2, 3$) are possible; the corresponding values of m are indicated. Obviously, by virtue of the fact that the system is 2π-periodic in t it is sufficient to consider only the two values $t^0 = 0$ and $t^0 = \pi$ and to construct only the two sets $\mathbf{M}_0^\pi$ and $\mathbf{M}_1^\pi$; $\mathbf{M}_1^\pi$ is the image of the set $\mathbf{M}_1 = \{\mathbf{u}, \upsilon : \upsilon = \pi\}$.

When investigating any specific mechanical problem we need to know the initial instant of time $t^0 = 0$ or $t^0 = \pi$. We then determine, using the above method, all the 2π-periodic motions beginning with oscillatory motions ($m = 0$) and increasing $m = 1, 2, \ldots$. We determine all 4π-periodic motions, 6π-periodic motions, etc. It can be seen from this simple scheme that the 2π-periodic motions contain such qualitatively different forms of motions as oscillations ($m = 0$), slow rotations ($|m| > 0$, $|m|$ is a small quantity), and rapid rotations ($|m| \gg 1$).

5. THE PROBLEM OF THE CONTINUATION WITH RESPECT TO A PARAMETER OF SYMMETRICAL PERIODIC MOTIONS OF A REVERSIBLE SYSTEM

We will consider a sufficiently smooth autonomous reversible system or one that is 2π-periodic with respect to t

$$\mathbf{u}^\cdot = \mathbf{U}(\mathbf{u}, \mathbf{v}, t) + \mu \mathbf{U}_1(\mu, \mathbf{u}, \mathbf{v}, t) \tag{5.1}$$

$$\boldsymbol{\upsilon}^\cdot = \mathbf{V}(\mathbf{u}, \mathbf{v}, t) + \mu \mathbf{V}_1(\mu, \mathbf{u}, \mathbf{v}, t); \quad \mathrm{u} \in \mathbb{R}^l$$

with a small parameter μ and a fixed set respectively $\mathbf{M} = \{\mathbf{u}, \mathbf{v} : \upsilon_i = 0\ (i = 1, \ldots, s \leqslant n), \sin \upsilon_j = 0\ (j = s + 1, \ldots, n)$ and $\mathbf{M} = \{t, \mathbf{u}, \mathbf{v} : \sin t = 0, \upsilon_i = 0\ (i = 1, \ldots, s \leqslant n)$, where it is assumed that system (5.1) is 2π-periodic with respect to the variable v_j ($j = s + 1, \ldots, n$).

Suppose that, when $\mu = 0$, in system (5.1) there is $2T^*$-periodic motion ($T^* = \pi k, k \in \mathbb{N}$, for a periodic system)

$$\mathrm{u} = \varphi(t), \ \boldsymbol{v} = \psi(t); \varphi(-t) = \varphi(t), \ \psi(-t) = -\psi(t) + 2\pi\boldsymbol{\beta}(\boldsymbol{\beta} \in \mathbb{Z}^n, \ \beta_1 = \ldots = \beta_s = 0) \tag{5.2}$$

symmetrical with respect to the fixed M. We will investigate the issue on the existence in system (4.1) when $\mu \neq 0$ of symmetrical, $2T$-periodic motions ($T = \pi k$ for a periodic system), which become the motion (5.2) when $\mu = 0$.

We will denote by $\mathbf{u}(\mu, \mathbf{u}^0, \mathbf{v}^0, t^0, t^0 + t)$, $\mathbf{v}(\mu, \mathbf{u}^0, \mathbf{v}^0, t^0, t^0 + t)$ the solution of system (5.1) with initial point $(\mathbf{u}^0, \mathbf{v}^0)$ when $t = 0$. Then, the necessary and sufficient conditions for symmetrical, $2T$-periodic motion to exist have the form (4.4) or (4.8) and represent a system of n functional equations in $u_1^0, \ldots, u_l^0$ (and T in the case of an autonomous system). This system has the solution $\mathbf{u} = \varphi(0)$ when $\mu = 0$. Hence, for sufficiently small $\mu \neq 0$ the system has a solution when the following condition is satisfied:

for a 2π-periodic system

$$\operatorname{rank}\left\|\frac{\partial v_\varkappa(\mu, \mathbf{u}^0, \pi\beta, \pi\alpha, \pi(\alpha+k))}{\partial u_\nu^0}\right\|_* = n \tag{5.3}$$

for an autonomous system

$$\operatorname{rank}\left\|\frac{\partial v_\varkappa(\mu, \mathbf{u}^0, \pi\beta, t^0, t^0+T)}{\partial u_\nu^0}, \frac{\partial v_\varkappa(\mu, \mathbf{u}^0, \pi\beta, t^0, t^0+T)}{\partial T}\right\| = n \tag{5.4}$$

The calculations in (5.3) and (5.4) are carried out for $\mu = 0$, $\mathbf{u}^0 = \varphi(0)$. Consequently, when condition (5.3) or (5.4) is satisfied the problem of existence when $\mu \neq 0$ in an autonomous or periodic system (5.1) is solved solely by the generating system obtained from (5.1) when $\mu = 0$. This case is a rough one.

Theorem 6. In the rough case (5.3) or (5.4) of periodic or autonomous system (5.1) when $\mu = 0$ we have an $(l-n)$- or $(l-n+1)$-parametric family of symmetrical periodic motions, which contain motion (5.2), and this family is continued in a unique way with respect to the parameter μ.

Condition (5.3) is only satisfied when $l \geqslant n$. If $l = n$, the generating system has a unique symmetrical, $2\pi k$-periodic motion, which is continued in a unique way with respect to μ. In autonomous system (5.1) when $l = n - 1$ and condition (5.4) is satisfied both when $\mu = 0$ and when $\mu \neq 0$, there is a unique symmetrical, $2T(\mu)$-periodic motion ($T(0) = T^*$). When $l \geqslant n$ two cases are possible in an autonomous system. In the first of these, among the first l columns of the matrix in (5.4) there are n linearly independent motions. Then, the generating system has an $(l - n + 1)$-family of $l - n$ quantities of $u_1^0, \ldots, u_l^0$ and T symmetrical, $2T$-periodic motions; this family is continued with respect to the parameter μ. Here, when $\mu \neq 0$, one cannot guarantee the existence of a symmetrical periodic motion of the same period as in the generating system.

It follows from the above that the problem of the continuation with respect to a parameter of a symmetrical periodic rotational motion in rough cases can be solved in the same way as in the special case of oscillatory motion [10, 11]. The same situation arises in non-rough cases. The theory for these cases was in fact described earlier [12, 13]. A system of the standard type is considered in [7].

6. A SYSTEM CLOSE TO A CONSERVATIVE SYSTEM WITH ONE DEGREE OF FREEDOM

Consider the equation

$$z^{\cdot\cdot} + f(z) = \mu F(\mu, z, z^\cdot, t), \quad \int_0^{2\pi} f(z)dz = 0 \tag{6.1}$$

where the functions $f(z)$, $F(\mu, z, z^\cdot, t)$ are 2π-periodic in z and t. When $\mu = 0$ Eq. (6.1) has the energy integral

$$z^{\cdot 2} + V(z) = x(\text{const}), \quad V(z) = 2\int f(z)dz$$

where we can assume that the 2π-periodic function $V(z)$ contains no constant term. Then, when $x > \max V(z)$ we have rotational motions (Fig. 3), which are closed on the cylinder $(z, z^\cdot)$. In the phase plane $(z, z^{\cdot})$ these solutions are not closed, and the velocity $z^{\cdot}$ keeps its sign. The period of the rotational motions is given by the formula

$$2T(x) = \int_0^{2\pi} \frac{dz}{g(x,z)}, \quad g(x,z) = \sqrt{x - V(z)}$$

The motions will therefore be $(2\pi k/|m|)$-periodic with respect to time $t (k \in \mathbb{N}, m \in \mathbb{Z})$, if the energy constant x satisfies the condition $T(x) = \pi k/|m|$.

The problem of the existence in (6.1) of oscillatory motions, which, when $\mu = 0$, convert into one of the oscillations (Fig. 3) of the generating system, was investigated in [14]. Below we will investigate the problem of the existence of $2\pi k$-periodic rotational motions of Eq. (6.1) when $\mu \neq 0$. Here, without loss of generality, we will consider rotations with positive velocity $z^{\cdot}$.

We will introduce new variables of the problem: x, y $(y = z)$. Then Eq. (6.1) can be written in the form of the system

$$x^{\cdot} = 2\mu g(x,y) F(\mu, y, g(x,y), t), \quad y^{\cdot} = g(x,y) \tag{6.2}$$

which is 2π-periodic in y and t. When $\mu = 0$, system (6.2) has a $2T(x^0)$-periodic rotational motion for any y^0 and $x^0 > \max V(z)$; x^0 and y^0 are the initial values of x and y when $t = t^0 = 0$. Consequently, we have the case considered in Section 3, Subsection 2.

Suppose $T(x^*) = \pi k/|m|$ and the function $\psi(x^*, y^0, t)$ satisfies the second equation in (6.2) when $x = x^*$. We then calculate

$$t = \int_{y^0}^{\psi(x^*, y^0, t)} \frac{d\xi}{g(x^*, \xi)}, \quad \frac{\partial \psi}{\partial x^*} = \frac{g(x^*; \psi)}{2} \int_{y^0}^{\psi(x^*, y^0, t)} \frac{d\xi}{[g(x^*, \xi)]^3} > 0$$

The condition $\partial\psi/\partial x^* \neq 0$ (when $t = 2\pi k$) is then satisfied, and the problem of the existence of $2\pi k$-periodic rotational motion when $\mu \neq 0$ is solved by the amplitude equation

$$I(y^0) = \int_0^{2\pi k} F(0, \psi(x^*, y^0, t), g(x^*, \psi(x^*, y^0, t), t) g(x^*, \psi(x^*, y^0, t)) dt = 0 \tag{6.3}$$

Theorem 7. To each simple root $y^0 = y^*$ of amplitude equation (6.3) there corresponds a unique $2\pi k$-periodic rotational motion

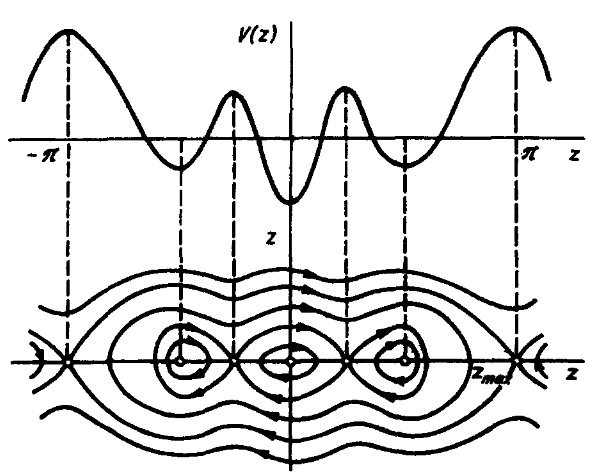

Fig. 3.

$$z = \psi(x^*, y^*, t) + O(\mu)$$

$$T(x^*) = \pi k / |m|, \quad \psi(x^*, y^*, t + 2\pi k) = \psi(x^*, y^*, t) + 2\pi m; \quad m \in \mathbb{Z}, m \neq 0$$

Note. In the conditions of Theorem 7 the variable x in (6.2) is defined by the expression

$$x = x^* + 2\mu \int_0^{\psi(x^*, y^*, t)} F(0, y, g(x^*, y)) dy + o(\mu)$$

In autonomous equation (6.1) the integral $I(y^0) \equiv 0$ and Theorem 7 is inapplicable. We will write Eq. (6.1) in the form

$$dx / dy = 2\mu F(\mu, y, g(x, y)) \tag{6.4}$$

Then, to each simple root $x = x^*$ of the amplitude equation

$$\int_0^{2\pi} F(0, y, g(x, y)) dy = 0$$

there corresponds [7] a unique solution of Eq. (6.4), 2π-periodic in y. Now, knowing from (6.4) the relationship $x(y)$ for the 2π-periodic solution, we can determine the function $z(t)$ from the second equation of system (6.2). As a result, the initial equation (6.1) allows of a single-parameter family of periodic rotational motions.

Note that it follows from the results obtained, in particular, that steady-state solutions of the averaged system when investigating rotational motions using Volosov's method [15], correspond to exact periodic rotational motions.

The reversible equation. We will assume that the functions f and F in (6.1) satisfy the additional conditions

$$f(-z) = -f(z), \quad F(\mu, -z, z^{\cdot}, -t) = -F(\mu, z, z^{\cdot}, t) \tag{6.5}$$

Then Eq. (6.1), (6.5) is reversible with a fixed set $\mathbf{M} = \{t, z, z^{\cdot} : \sin t = 0, \sin z = 0\}$. Obviously, in this case the generating equation is also reversible with a fixed set $\mathbf{M}^* = \{z, z^{\cdot} : \sin z = 0\}$, and all the rotational motions, symmetrical with respect to M^*, are described by odd functions $t - t^0$ (t^0 is the initial instant of time). The set $\{t, x, y : \sin t = 0, \sin y = 0\}$ is a fixed set for reversible system (6.2), (6.5). Hence, the symmetrical, $2\pi k$-periodic rotational motion $\psi(x^*, \pi\beta, t)$, $\beta \in \mathbb{Z}$ $T(x^*) = \pi k | m |$ ($k \in \mathbb{N}$, $m \in \mathbb{Z}$) of the generating system for (6.2) is continued with respect to the parameter μ if (Theorem 6)

$$\partial\psi(x^*, \pi\beta, \pi(\alpha + k))/\partial x^* \neq 0$$

($\pi\alpha$ is the initial value of t, in which case $z = \pi\beta$). It was shown above that this condition is always satisfied.

Theorem 8. For the reversible equation (6.1), (6.5) all $2\pi k$-periodic rotational motions

$$T(x^*) = \pi k/|m|, \psi(x^*, \pi\beta, t + 2\pi k) = \psi(x^*, \pi\beta, t) + 2\pi m; \ k \in \mathbb{N}, m, \beta \in \mathbb{Z}$$

are continued with respect to the parameter μ.

In the autonomous case, when conditions (6.5) are satisfied, Eq. (6.4) describes motion in an invariant set. Consequently [5], all the solutions of Eq. (6.1), (6.5) are 2π-periodic in z. In particular, this can be constant solutions.

In view of the fact that the function $V(z)$ is even, the zero will be the equilibrium position for the generating equation (Fig. 3). It follows from the fact that $V(z)$ is 2π-periodic that the points $z = \pi\beta$ ($\beta \in \mathbb{Z}$) are also equilibria. We will consider oscillatory motions of only one of these equilibria (for example, the point $z = 0$) and we will investigate the problem of the continuation of the $2\pi k$-periodic oscillations with respect to the parameter μ. Note that the oscillatory motion considered may contain an odd number of equilibrium positions inside it (Fig. 3).

Suppose $z(z_0', t)$ is an oscillatory solution of Eq. (6.1), (6.5) when $\mu = 0$ with initial point $z = 0, z' = z_0'$ when $t = 0$. We will calculate $\partial z/\partial z_0'$ when $T = \pi k$. When $t > T/2$ we have (Fig. 3)

$$\int_0^z \frac{d\xi}{g(x^*,\xi)} = T - t, \quad T(x^*) = 2\int_0^{z_{max}} \frac{d\xi}{g(x^*,\xi)}$$

Now, taking into account the fact that $x^* = z_0'^2$, we have

$$\frac{1}{g(x^*,z)}\frac{\partial z}{\partial z_0'} - z_0'\int_0^z \frac{d\xi}{[g(x^*,\xi)]^3} = \frac{\partial T(x^*)}{\partial z_0'}$$

Since we have $z = 0$ when $t = T$, we obtain

$$\partial z/\partial z_0 |_{t=T} = z_0' \partial T(x^*)/\partial z_0$$

where $z_0' \neq 0$.

Theorem 9. The symmetrical, $2\pi k$-periodic ($k \in \mathbb{N}$) oscillatory motion of Eq. (6.1), (6.5) is continued with respect to the parameter μ, if

$$T(x^*) = \pi k/|m|, \; m \in \mathbb{Z}, \; \partial T(x^*)/\partial x^* \neq 0$$

7. A SATELLITE IN THE PLANE OF AN ELLIPTIC ORBIT

The motion of a satellite in this problem is described by the equation [16]

$$\alpha'' - \frac{2e\sin\upsilon}{1+e\cos\upsilon}(\alpha'+1) + \frac{\mu\sin\alpha\cos\alpha}{1+e\cos\upsilon} = \varepsilon F(\varepsilon,\alpha,\alpha',\upsilon), \varepsilon \ll 1 \tag{7.1}$$

Here e is the eccentricity of the elliptic orbit ($0 \leqslant e < 1$), μ is the inertial parameter ($|\mu| \leqslant 3$), α is the angle between the radius vector of the centre of mass and the principle central axis of inertia in the plane of the orbit, υ is the true anomaly, chosen as the independent variable, and ε is a small parameter.

When $\varepsilon = 0$ we have Beletskii's equation [17], which describes the motion of a satellite due solely to gravitational forces. The perturbations εF are due to the action of the atmosphere, light pressure, the magnetic field and other factors, ignored in the model. We will assume that the function F is 2π-periodic in α, and υ.

Equation (7.1) has been investigated in numerous publications (see [16]). The most complete results have been obtained for $\varepsilon = 0$. Here all possible odd oscillatory and rotational motions have been constructed and their stability has been investigated [2, 16]. The problem of the motion of a satellite acted upon by gravitational forces and the resistance of the atmosphere [16] and the problem of the motion under gravitational forces and light pressure [18, 19] have been considered separately. In the latter problem a systematic study of periodic rotational motions was begun in [19].

When $\varepsilon = 0$, Eq. (7.1) is reversible with a fixed set $\mathbf{M} = \{\upsilon, \alpha, \alpha': \sin\upsilon = 0, \sin\alpha = 0\}$. This immediately follows from the invariance of Eq. (7.1) when $\varepsilon = 0$ with respect to the replacement $(\upsilon, \alpha, \alpha') \to (-\upsilon, -\alpha, \alpha')$ and the fact that it is 2π-periodic in υ and α. Moreover, when α is replaced by $\alpha + \pi/2$, Beletskii's equation retains its form when μ is replaced by $-\mu$ and remains reversible.

1. *A satellite in a slightly elliptic orbit* ($e \ll 1$). When $e = 0$ and $\varepsilon = 0$ Eq. (7.1) describes the motion of a mathematical pendulum and has the energy integral

$$\alpha'^2 + \mu\sin^2\alpha = h(h = \text{const})$$

When $\mu > 0$ the zeroth equilibrium position is stable and oscillatory motion exist around it. The period of these oscillations is

$$2T = \frac{2}{\sqrt{\mu}}\int_0^1 \frac{du}{\sqrt{(1-u^2)(1-\varkappa^2u^2)}}, \quad \mu\varkappa^2 = \alpha_0'^2$$

(α_0' is the initial velocity $\alpha = \alpha_0 = 0$). Hence $dT/d\varkappa \neq 0$, and all the $2\pi k$-periodic oscillations for which $T = \pi k/m$ ($k, m \in \mathbb{N}$) are continued with respect to the parameters e and ε if the perturbations belong to the class of reversible perturbations.

The case $\mu < 0$ can be reduced to the case when $\mu > 0$ by replacing α by $\alpha + \pi/2$.

Theorem 10. If $\mu > 0$, all $2\pi k$-periodic oscillatory motions of the satellite are continued with respect to the parameters e and ε, if the perturbations satisfy the condition

$$F(\varepsilon, -\alpha, \alpha', -\upsilon) = -F(\varepsilon, \alpha, \alpha', \upsilon) \tag{7.2}$$

When $\mu < 0$ the similar condition takes the form

$$F(\varepsilon, -\alpha + \pi/2, \alpha', -\upsilon) = -F(\varepsilon, \alpha + \pi/2, \alpha', \upsilon) \tag{7.3}$$

The following assertion follows from Theorem 8.

Theorem 11. For sufficiently small $\varepsilon \neq 0$ in a slightly elliptical orbit ($e \ll 1$) there are $2\pi k$-periodic ($k \in \mathbb{N}$) rotational motions of the satellite produced from the rotational motions of the satellite in a circular orbit with period

$$2T = \int_0^{2\pi} \frac{d\xi}{\sqrt{h - \mu \sin^2 \xi}} = \frac{2\pi k}{|m|}, \quad m \in \mathbb{Z}$$

provided the perturbations satisfy condition (7.2) or (7.3).

Rotational motions occur when $\mu < 0$ ($\mu > 0$), if the energy $h > 0$ ($h > \mu$). In these motions, the satellite turns in relative motion by $|m|$ rotations per k rotations of the centre of mass in the elliptic orbit. When $m > 0$ the satellite rotates in the same direction as the centre of mass (direct rotation) and when $m < 0$ we have the opposite rotation. If $m = -1$, we have an "almost equilibrium" orientation of the satellite in an absolute system of coordinates.

2. *A satellite close to a dynamically symmetrical body*. When $\mu = 0$, $\varepsilon = 0$ (a dynamically symmetrical satellite in the Beletskii problem) we have

$$\alpha'' - \frac{2e \sin \upsilon}{1 + e \cos \upsilon}(\alpha' + 1) = 0 \tag{7.4}$$

The general solution of this equation is given by the formula

$$\alpha = -\upsilon + (\alpha_0' + 1) \int_0^{\upsilon} \frac{(1+e)^2 d\upsilon}{(1 + e\cos\upsilon)^2} + \alpha_0$$

(α_0 and α_0^1 are the values of the angle and angular velocity at the perigee of the orbit) and represents uniform rotation with angular velocity $\alpha_0' + 1$ in an absolute system of coordinates. This solution will be $2\pi k$-periodic rotation if

$$\alpha' = -1 + (m/k + 1)(1 - e)^{3/2}(1 + e)^{-1/2}, \quad m \in \mathbb{Z} \tag{7.5}$$

These curves are shown in Fig. 4 when $k = 1$.

If $\alpha_0 = 0, \pm \pi/2, \pm\pi, \ldots$, these solutions are symmetrical with respect to a fixed set M. Hence, a sufficient condition for the continuation of these rotational motions with respect to the parameters μ and ε is the inequality (Theorem 6)

$$\left.\frac{d\alpha}{\partial \alpha_0'}\right|_{\upsilon = \pi k} = \int_0^{\pi k} \frac{(1+e)^2 d\upsilon}{(1 + e\cos\upsilon)^2} \neq 0$$

Theorem 12. In problem (7.1) with perturbations (7.2) or (7.3) the satellite, close to dynamically symmetrical body, executes $2\pi k$-periodic rotations, close to uniform rotations with angular velocity (7.5)

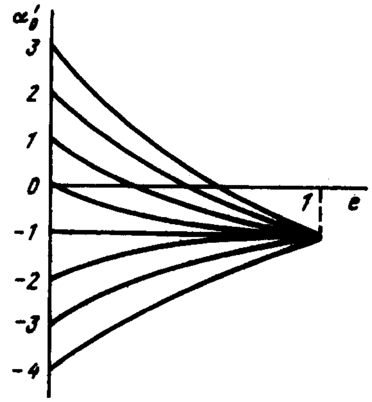

Fig. 4.

both in the forward direction ($m > 0$) and in the reverse direction ($m < 0$). Here, after k rotations of the centre of mass in an elliptic orbit the satellite makes $|m|$ rotations around the centre of mass.

Hence, it follows that the resonance rotations of the satellite in Beletskii's problem, established in [20] by an asymptotic method, correspond to precise periodic rotations.

3. *Beletskii's perturbed problem*. When $\varepsilon = 0$ odd $2\pi k$-oscillatory motions of the satellite have been constructed (see [16]) and their stability has been investigated in the linear approximation. It follows from these results that the limit of the region of stability forms a set of zero measure in the (α_0', e) plane. Hence we can conclude (Theorem 2) that almost all these oscillatory motions are continued with respect to the parameter ε. In the class of perturbations (7.2) or (7.3), Eq. (7.1) remains reversible. Hence, the pure imaginary characteristic exponents for the oscillatory motions when $\varepsilon = 0$ remain pure imaginary when $|\varepsilon| \neq 0$ is sufficiently small, provided there are no second-order resonances [21]. With the additional condition that there are no resonances of up to the fourth order inclusive, the Lyapunov stability condition is specified [22] by the difference from zero of one coefficient $C(\alpha_0', \mu, e)$ in fourth-order forms of the equations of perturbed motion. This coefficient depends on (α_0', μ, e), and the condition $C(\alpha_0', \mu, e)$ distinguishes a set of zero measure in the region of stability in the first approximation in the (α_0', e) plane.

Note that in Beletskii's problem an investigation of the stability of oscillatory motions in a strict non-linear formulation based on the Arnol'd–Moser theorem [23, 24] and the theorem on the stability of a periodic Hamilton system with one degree of freedom [25] was carried out in [26].

All the initial velocities $\alpha^1{}_0$ for rotational $2\pi k$-periodic motions when $\varepsilon = 0$ can be constructed numerically using a method based on Theorem 5. These results were obtained previously in [2, 19]. The characteristic exponents are determined by constructing [27] only one partial solution of the system in variations. Further, from the discussion in Sections 2 and 5 and also above in this section one can derive the correctness of the assertions regarding the continuation of rotational motions with respect to the parameter ε and their Lyapunov stability.

Theorem 13. Almost all $2\pi k$-periodic oscillatory and rotational motions in the Beletskii problem are continued with respect to the small parameter ε. Almost all $2\pi k$-periodic oscillatory and rotational motions in Beletskii's problem, that are stable in the linear approximation, are Lyapunov stable and almost all these motions, when $\varepsilon \neq 0$ in the class of reversible perturbations (7.2) or (7.3), lead to Lyapunov-stable oscillatory and rotational motions of period $2\pi k$.

This research was supported financially by the Russian Foundation for Basic Research (96-15-96051 and 97-01-00538).

REFERENCES

1. SADOV, Yu. A. Periodic motions of a satellite with a magnetic damper in the plane of a circular orbit. *Kosmich. Issled.*, 1969, **7**, 1, 51–60.
2. SARYCHEV, V. A., SAZONOV, V. V. and ZLATOUSTOV, V. A., Periodic rotations of a satellite in the plane of an elliptic orbit. *Kosmich Issled.*, 1979, **17**, 2, 190–207.

3. POINCARÉ, H., *Les methodes nouvelles de la mecanique céleste*. 1892–1899.
4. ANDRONOV, A. A. and VITT, A. A., On Lyapunov stability. *Zh. Eksp. Teor. Fiz.*, 1933, **2**, 5, 373–374.
5. MALKIN, I. G., *Theory of the Stability of Motion*. Nauka, Moscow, 1966.
6. MALKIN, I. G., *Some Problems of the Theory of Non-linear Oscillations*. Gostekhizdat, Moscow, 1956.
7. TKHAI, V. N., The Lyapunov–Poincaré method in the theory of periodic motions. *Prikl. Mat. Mekh.*, 1998, **62**, 3, 355–371.
8. HEINBOCKEL, J. H. and STRUBLE, R. A., Periodic solutions for differential systems with symmetries. *J. Soc. Indust. Appl. Math.*, 1965, **13**, 2, 425–440.
9. TKHAI, V. N., Invariant sets and symmetric periodic motions of reversible mechanical systems. *Prikl. Mat. Mekh.*, 1996, **60**, 6, 959–972.
10. TKHAI, V. N., Non-linear oscillations of reversible systems. *Prikl. Mat. Mekh.*, 1995, **59**, 1, 38–50.
11. TKHAI, V. N., Symmetric periodic orbits of the many-body problem. Resonance and parade of planets. *Prikl. Mat. Mekh.*, 1995, **59**, 3, 355–364.
12. TKHAI, V. N., Symmetric periodic orbits in the restricted three-body problem. *Kosmich. Issled.*, 1997, **35**, 2, 164–171.
13. TKHAI, V. N., Continuation of periodic motions of a reversible system in non-structurally stable cases. Application to the *N*-planet problem. *Prikl. Mat. Mekh.*, 1998, **62**, 1, 56–72.
14. KATS, A. M., Forced oscillations of non-linear systems with one degree of freedom, close to conservative. *Prikl. Mat. Mekh.*, 1995, **19**, 1, 13–32.
15. VOLOSOV, V. M. and MORGUNOV, B. I., *The Method of Averaging in the Theory of Non-linear Oscillatory Systems*. Izd. MGU, Moscow, 1971.
16. SARYCHEV, V. A., Problems of the orientation of artificial satellites. *Results of Science and Technology, Series Space Research*, Vol. 11. Vsesoyuz. Nauch. i Tekh. Inform. Moscow, 1978.
17. BELETSKII, V. V., The motion of an artificial Earth satellite about a centre of mass. In *Artificial Earth Satellites*. Izd. Akad. Nauk SSSR, Moscow, 1958, 1, 25–43.
18. FLANAGAN, R. C. and MODI, V. J., Attitude dynamics of a gravity orientated satellite under the influence of solar radiation pressure. *Aeronaut. J.*, 1970, **74**, 718, 835–841.
19. GRODMAN, D. L. and TKHAI, V. N., A numerical analysis of the periodic motions of a satellite in an elliptic orbit under the action of gravitational forces and light pressure. In *Problems of the Investigation of the Stability and Stabilization of Motion*. Computing Centre, Russian Academy of Sciences, Moscow, 1998, 107–116.
20. CHERNOUS'KO, F. L., Resonance phenomena when a satellite moves about a centre of mass. *Zh. Vychisl. Mat. Mat. Fiz.*, 1963, **3**, 3, 528–538.
21. TKHAI, V. N., Some problems on the stability of reversible systems with a small parameter. *Prikl. Mat. Mekh.*, 1994, **58**, 1, 3–12.
22. BIBIKOV, Yu. N., *Multifrequency Non-linear Oscillations and their Bifurcation*. Izd. LGU, Leningrad, 1991.
23. ARNOL'D, V. I., The stability of the equilibrium positions of a Hamiltonian system of ordinary differential equations in the general elliptic case. *Dokl. Akad. Nauk SSSR*, 1961, **137**, 2, 255–257.
24. MOSER, J., On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Gottingen. Math. Phys.* 1962, **11a**, 1, 1–20.
25. MARKEYEV, A. P., *Libration Points in Celestial Mechanics and Space Dynamics*. Nauka, Moscow, 1978.
26. ZLATOUSTOV, V. A. and MARKEYEV, A. P., Stability of planar oscillations of a satellite in an elliptic orbit. *Celest. Mech.*, 1973, **7**, 1, 31–45.
27. ZIMOVSHCHIKOV, A. S. and TKHAI, V. N., The stability of triangular solutions of the unrestricted three-body problem. *Problems of the Investigation of the Stability and Stabilization of Motion*. Computing Centre Russian Academy of Sciences, Moscow, 1998, 117–130.

Translated by R.C.G.